On singular perturbation problems of non-linear systems of differential equations.

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1. Introduction. We consider a non-linear system of differential equations containing a real parameter ε

$$(1)\qquad \begin{aligned} \frac{dx_i}{dt} &= f_i(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n, t; \varepsilon) \quad (i=1, 2, \ldots, m), \\ \varepsilon\frac{dx_j}{dt} &= f_j(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n, t; \varepsilon) \quad (j=m+1, m+2, \ldots, n), \end{aligned}$$

where $f_k(x_1, \ldots, x_n, t; \varepsilon)$ $(k=1, 2, \ldots, n)$ are continuous functions of all the arguments $x_1, \ldots, x_n, t$, and ε is a small parameter. When ε is zero, the system (1) is reduced to the so-called degenerate system

$$(2)\qquad \begin{aligned} \frac{dx_i}{dt} &= f_i(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n, t; 0) \quad (i=1, 2, \ldots, m), \\ 0 &= f_j(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n, t; 0) \quad (j=m+1, \ldots, n). \end{aligned}$$

In the singular perturbation theory, many authors have discussed the relationship between the solutions of the system (1) and those of the degenerate system (2) as ε tends to zero, cf., for example, [1], [2]. In these studies the differentiability of the right members of the system is assumed, and particularly the differentiability of $f_j(x_1, \ldots, x_{m+1}, \ldots, x_n, t, 0)$ $(j=m+1, \ldots, n)$ with respect to $x_k (k=m+1, \ldots, n)$ seems to be essential. On the other hand, Prof. M. Nagumo [3] treated of a differential equation of the second order

$$\lambda\frac{d^2y}{dx^2} + f\left(x, y, \frac{dy}{dx}, \lambda\right) = 0 \quad \lambda > 0,$$

and studied the behavior of the solution of this equation as λ tends to zero. In his work, he does not assume the differentiability of the function $f(x, u, v, 0)$ with respect to v, but an inequality

$$\frac{f(x, u, \bar{v}, 0) - f(x, u, v, 0)}{\bar{v} - v} > L \quad (>0)$$

for any two points (x, u, v), $(x, u, \bar{v})$ belonging to the domain of the definition of the function $f(x, u, v, 0)$.

In the present paper, we discuss, by using Nagumo's idea, the singular perturbation problems of the systems (1) and (2) under weaker hypotheses than in the papers referred in this paragraph.

2. Main theorem. We first assume the following condition.

C1: The degenerate system (2) possesses a solution $x_k=X_k(t)$ $(k=1,2,\dots,n)$, which is continuously differentiable on a finite closed interval $a\leqq t\leqq a+l$.

A transformation $t=\tau+a$ permits us to suppose $a=0$ without loss of generality.

In the sequel, we suppose that the functions $f_k(x_1,\dots,x_n,t;\varepsilon)$ $(k=1,2,\dots,n)$ are continuous in a domain

D: $|x_k-X_k(t)|\leqq d$ $(k=1,2,\dots,n)$, $0\leqq t\leqq l$,

where $0<d$, $0<l<\infty$, and we consider only the non-negative values of the small parameter ε.

Let the functions $f_k(x_1,\dots,x_n,t;\varepsilon)$ fulfil the following conditions.

C2: The functions $f_j(x_1,\dots,x_n;t;0)$ $(i=1,2\dots,m)$ satisfy a Lipschitz condition with respect to all the arguments $x_1,\dots,x_n$, i. e. for any two points $(\bar{x}_1,\dots,\bar{x}_n,t)$, $(x_i,\dots,x_n,t)\in D$, we have

$|f_i(\bar{x}_1,\dots,\bar{x}_n,t,0)-f_i(v_1,\dots,x_n,t,0)|\leqq K_1\sum_{k=1}^{n}|\bar{x}_k-x_k|$.

C3: The functions $f_j(x_1,\dots,x_m,x_{m+1},\dots,x_n,t;0)$ $(j=m+1,\dots,n)$ satisfy a Lipschitz condition with respect to the arguments $x_1,\dots,x_m$, i.e. for any two points $(\bar{x}_1,\dots,\bar{x}_m,x_{m+1},\dots,x_n,t)$, $(x_1,\dots,x_m,x_{m+1},\dots,x_n,t)\in D$, we have $|f_j(x_1,\dots,x_m,x_{m+1},\dots,x_n,t;0)-f_j(x_1,\dots,x_m,x_{m+1},\dots,x,t;0)|\leqq K_2\sum_{i=1}^{m}|\bar{x}_i-x_i|$.

C4: For any $j(j=m+1,\dots,n)$, the function $f_j(x_1,\dots,x_m,x_{m+1},\dots,x_n,t;0)$ does not depend on the arguments $x_{m+1},\dots,x_{j-1},x_{j+1},\dots,x_n$, and we have

$$\frac{f_j(x_1,\dots,x_{j-1},\bar{x}_j,x_{j+1},\dots,x_n,t;0)-f_j(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_n,t;0)}{\bar{x}_j-x_j}<-L(<0),$$

for $(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_n,t)$, $(x_1,\dots,x_{j-1},\bar{x}_j,x_{j+1},\dots,x_n,t)\in D$.

C5: The functions $f_k(x_1,\dots,x_n,t;\varepsilon)$ $(k=1,2,\dots,n)$ converge uniformly to the functions $f_k(x_1,\dots,x_n,t;0)$ respectively in the domain D as ε tends tozero. This condition implies that there exists a function of ε, $\omega(\varepsilon)$, such that

$\operatorname*{Max}_{D}|f_k(x_1,\dots,x_n,t;\varepsilon)-f_k(x_1,\dots,x_n,t;0)|<\omega(\varepsilon)$

and $\omega(\varepsilon)$ tends to zero as ε tends to zero.

THEOREM 1. *Given n functions of ε, $h_k(\varepsilon)$ $(k=1,2,\dots,n)$, which tend to zero with ε. Under the conditions* C1, C2, C3, C4 *and* C5, *the non-linear system of differential equations* (1) *possesses a solution $x_k(t;\varepsilon)$ $(k=1,2,\dots,n)$ over the interval $0\leqq t\leqq l$, which satisfies the initial condition*

$$x_k(0;\varepsilon)=X_k(0)+h_k(\varepsilon)\ (k=1, 2, \dots, n)\ ,$$

provided that ε *is sufficiently small.*

Moreover, the solution $x_k(t;\varepsilon)$ *converges uniformly to the solution* $X_k(t)$ *over the interval* $0\leqq t\leqq l$ *as* ε *tends to zero.*

3. **Lemma.** To prove the Theorem 1, we use the following lemma.

LEMMA. *Suppose that the functions* $f_k(x_1,\dots,x_n,t)\ (k=1,2,\dots,n)$ *are defined and continuous in a domain*
B: $\underline{\alpha}_k(t)\leqq u_k\leqq\overline{\alpha}_k(t)\ (k=1, 2, \dots, n),\qquad 0\leqq t\leqq l$,
where $\underline{\alpha}_k(t)$, $\overline{\alpha}_k(t)$ *are continuously differentiable on the interval* $0\leqq t\leqq l$, *and satisfy the following differential inequalities:*

$$\overline{\alpha}_k'(t)>f_k(u_1,\dots,u_{k-1},\overline{\alpha}_k(t),u_{k+1},\dots,u_n,t)$$

$$\underline{\alpha}_k'(t)<f_k(u_1,\dots,u_{k-1},\underline{\alpha}_k(t),u_{k+1},\dots,u_n,t)$$

for $\underline{\alpha}_j(t)\leqq u_j\leqq\overline{\alpha}_j(t)\ (j\neq k),\ (1\leqq k\leqq n)$.

Under these conditions we consider a system of differential equations

$$\frac{du_k}{dt}=f_k(u_1,\dots,u_n,t)\ (k=1, 2, \dots, n)\ .$$

Then there is a solution $u_k(t)$ *of this system, fulfilling the initial condition*

(I) $$\underline{\alpha}_k(0)\leqq u_k(0)\leqq\overline{\alpha}_k(0),$$

which is defined over the interval $0\leqq t\leqq l$, *and satisfies the inequalities*

(II) $$\underline{\alpha}_k(t)<u_k(t)<\overline{\alpha}_k(t)\ (k=1, 2, \dots, n)$$

on the interval $0<t\leqq l$.

PROOF. We see easily by a fundamental theorem of the theory of differential equations that there exists in a neighborhood of 0 a solution $u_k(t)$ of the considered system of differential equations satisfying the initial condition (I), and that there exists a positive number δ such that for $0<t<\delta$

$$\underline{\alpha}_k(t)<u_k(t)<\overline{\alpha}_k(t)\ (i=1, 2, \dots, n)\ .$$

If the inequalities (II) do not hold on the interval $0<t\leqq l$, there exist a point ξ in the interval $0<t\leqq l$ and an index κ such that the inequalities (II) hold for $0<t<\xi$ and such that

(i) $\underline{\alpha}_\kappa(\xi)=u_\kappa(\xi)$, or (ii) $\overline{\alpha}_\kappa(\xi)=u_\kappa(\xi)$.

If the equality (i) holds, then we have

$$\underline{\alpha}_\kappa'(\xi)\geqq u_\kappa'(\xi)=f_\kappa(u_1,\dots,u_{\kappa-1},\underline{\alpha}_\kappa(\xi),u_{\kappa+1},\dots,u_n,t)\ ,$$

which is impossible. With the same reason, the equality (ii) can not hold. Therefore, we have the inequalities (II) on the interval $0<t\leqq l$.

4. Proof of Theorem 1. We put

$$x_k(t;\varepsilon)=X_k(t)+u_k(t;\varepsilon)\ (k=1,2,\dots,n)\ ,$$
$$F_i(u_1,\dots,u_n,t;\varepsilon)=f_i(X_1(t)+u_1,\dots,X_n(t)+u_n,t;\varepsilon)$$
$$-f_i(X_1(t),\dots,X_n(t),t;0)\ (i=1,2,\dots,m)\ ,$$

and

$$F_j(u_1,\dots,u_n,t;\varepsilon)=\frac{1}{\varepsilon}f_j(X_1(t)+u_1,\dots,X_n(t)+u_n,t;\varepsilon)$$
$$-\frac{dX_j}{dt}\quad(j=m+1,\dots,n)\ .$$

Then, the system of differential equations (1), is transformed into the following system

$$\frac{du_k}{dt}=F_k(u_1,\dots,u_n,t:\varepsilon)\ (k=1,2,\dots,n)\ , \tag{3}$$

where the functions $F_k(u_1,\dots,u_n,t;\varepsilon)$ are defined and continuous in a domain

$$D_1:\qquad |u_k|\leqq d\quad(k=1,2,\dots,n)\ ,\qquad 0\leqq t\leqq l\ .$$

Hence it is sufficient to show the following:

For sufficiently small $\varepsilon>0$, there exists a solution $u_k(t;\varepsilon)$ of the system (3) satisfying the initial condition

$$u_k(0;\varepsilon)=h_k(\varepsilon)\ (k=1,2,\dots,n)\ .$$

Furthermore, the solution $u_k(t;\varepsilon)$ exists over the interval $0\leqq t\leqq l$, and converges uniformly to zero as ε tends to zero.

Now, in order to apply the Lemma to the system (3), we will estimate the functions $F_k(u_1,\dots,u_n,t;\varepsilon)$. When $(u_1,\dots,u_n,t)\in D_1$, we have, for $i=1,2,\dots,m$,

$$\begin{aligned}
&|F_i(u_1,\dots,u_n,t;\varepsilon)|\\
&\quad=|f_i(X_1+u_1,\dots,X_n+u_n,t;\varepsilon)-f_i(X_1,\dots,X_n,t;0)|\\
&\quad\leqq|f_i(X_1+u_1,\dots,X_n+u_n,t;0)-f_i(X_1,\dots,X_n,t;0)|\\
&\quad+|f_i(X_1+u_1,\dots,X_n+u_n,t;\varepsilon)-f_i(X_1+u_1,\dots,X_n+u_n,t;0)|\\
&\quad<K_1\sum_{k=1}^{n}|u_k|+\omega(\varepsilon)\ ,
\end{aligned}$$

hence

$$(4)\qquad -K_1\sum_{k=1}^{n}|u_k|-\omega(\varepsilon)<F_i(u_1,\dots,u_n,t;\varepsilon)$$
$$<K_1\sum_{k=1}^{n}|u_k|+\omega(\varepsilon)\qquad (i=1,2\dots,m)\,.$$

Besides, we have, for $j=m+1,\dots,n$,

$$\begin{aligned}
&F_j(u_1,\dots,u_n,t;\varepsilon)\\
&=\frac{1}{\varepsilon}\{f_j(X_1+u_1,\dots,X_m+u_m,\dots,X_j+u_j,\dots,X_n+u_n,t;\varepsilon)\\
&\quad -f_j(X_1,\dots,X_m,\dots,X_j,\dots,X_n,t;0)\}-\frac{dX_j}{dt}\\
&=\frac{1}{\varepsilon}[\{f_j(X_1+u_1,\dots,X_n+u_n,t;\varepsilon)-f_j(X_1+u_1,\dots,X_n+u_n,t;0)\}\\
&\quad +\{f_j(X_1+u_1,\dots,X_m+u_m,\dots,X_j+u_j,\dots,X_n+u_n,t;0)\\
&\quad -f_j(X_1+u_1,\dots,X_m+u_m,X_{m+1},\dots,X_j,\dots,X_n,t;0)\}\\
&\quad +\{f_j(X_1+u_1,\dots,X_m+u_m,X_{m+1},\dots,X_n,t;0)\\
&\quad -f_j(X_1,\dots,X_m,\dots,X_n,t;0)\}]-\frac{dX_j}{dt}\,.
\end{aligned}$$

Therefore,

$$(5)\quad \begin{aligned}
&F_j(u_1,\dots,u_n,t;\varepsilon)<\frac{1}{\varepsilon}\{K_2\sum_{k=1}^{m}|u_k|-Lu_j+\omega(\varepsilon)\}+M\quad\text{for}\quad u_j\geqq 0\,,\\
&F_j(u_1,\dots,u_n,t;\varepsilon)>\frac{1}{\varepsilon}\{-K_2\sum_{k=1}^{m}|u_k|-Lu_j-\omega(\varepsilon)\}-M\quad\text{for}\quad u_j\leqq 0\,,
\end{aligned}$$

where $M\geq\underset{0\leqq t\leqq l}{\mathrm{Max}}\left|\frac{dX_j}{dt}\right|$ $(j=m+1,\dots,n)$.

We now consider a linear system of differential equations

$$(6)\quad \begin{aligned}
&\frac{dU_i}{dt}=K\sum_{k=1}^{n}U_k+\omega(\varepsilon)\qquad (i=1,2,\dots,m)\,,\\
&\frac{dU_j}{dt}=\frac{K}{\varepsilon}\sum_{k=1}^{m}U_k-\frac{L}{\varepsilon}U_j+\frac{\omega(\varepsilon)}{\varepsilon}+M\,,\qquad (j=m+1,\dots,n)
\end{aligned}$$

where $K\geq\mathrm{Max}\,\{K_1,K_2\}$.

Let us suppose that the system (6) possesses a solution $U_k(t;\varepsilon)$ satisfying the initial condition

$$|h_k(\varepsilon)|\leqq U_k(0;\varepsilon)\qquad (k=1,2,\dots,n)\,,$$

and that these functions $U_k(t;\varepsilon)$ are positive and $U_k(t;\varepsilon)<d$ on the entire

interval $0 \leqq t \leqq l$. We put

$$\bar{\alpha}_k(t;\varepsilon)=U_k(t;\varepsilon)\,,\quad \underline{\alpha}_k(t;\varepsilon)=-U_k(t;\varepsilon)\quad (k=1,2,\dots,n)\,.$$

Thus we obtain from (4), for $i=1,2,\dots,m$,

$$\begin{aligned}\bar{\alpha}_i'(t;\varepsilon)&=K\sum_{k=1}^{n}U_k(t;\varepsilon)+\omega(\varepsilon)\\&=K\sum_{k=1}^{n}\bar{\alpha}_k(t;\varepsilon)+\omega(\varepsilon)\\&\geqq K\{\bar{\alpha}_i(t;\varepsilon)+\sum_{k\neq i}|u_k|\}+\omega(\varepsilon)\\&>F_i(u_1,\dots,u_{i-1}\,,\quad \bar{\alpha}_i(t;\varepsilon),u_{i+1},\dots,u_n;t;\varepsilon)\,,\end{aligned}$$

provided $|u_k|\leqq\bar{\alpha}_k(t;\varepsilon)\,(k\neq i)$. By using (5), we have, for $j=m+1,\dots,n$,

$$\begin{aligned}\bar{\alpha}_j'(t;\varepsilon)&=\frac{K}{\varepsilon}\sum_{k=1}^{n}U_k(t;\varepsilon)-\frac{L}{\varepsilon}U_j(t;\varepsilon)+\frac{\omega(\varepsilon)}{\varepsilon}+M\\&=\frac{K}{\varepsilon}\sum_{k=1}^{n}\bar{\alpha}_k(t;\varepsilon)-\frac{L}{\varepsilon}\bar{\alpha}_j(t;\varepsilon)+\frac{\omega(\varepsilon)}{\varepsilon}+\mathrm{M}\\&\geqq\frac{K}{\varepsilon}\sum_{k=1}^{m}|u_k|-\frac{L}{\varepsilon}\bar{\alpha}_j(t;\varepsilon)+\frac{\omega(\varepsilon)}{\varepsilon}+M\\&>F_j(u_1,\dots,u_m,u_{m+1},\dots,u_{j-1},\bar{\alpha}_j(t;\varepsilon),u_{j+1},\dots,u_n,t;\varepsilon)\end{aligned}$$

provided $|u_k|\leqq\bar{\alpha}_k(t;\varepsilon)(j\neq k)$.

We have, for $i=1,2,\dots,m$,

$$\begin{aligned}\underline{\alpha}_i'(t;\varepsilon)&=-U_i'(t;\varepsilon)=-K\sum_{k=1}^{n}U_k(t;\varepsilon)-\omega(\varepsilon)\\&=K\sum_{k=1}^{n}\underline{\alpha}_k(t;\varepsilon)-\omega(\varepsilon)\\&=K\{\underline{\alpha}_i(t)+\sum_{k\neq i}(-|u_k|)\}-\omega(\varepsilon)\\&\leqq-K\{|\underline{\alpha}_i(t)|+\sum_{k\neq i}|u_k|\}-\omega(\varepsilon)\\&<F_i(u_1,\dots,u_{i-1},\underline{\alpha}_i(t;\varepsilon),u_{i+1},\dots,u_n,t;\varepsilon)\,,\end{aligned}$$

provided $|u_k|\leqq|\underline{\alpha}_k(t;\varepsilon)|=\bar{\alpha}_k(t;\varepsilon)(i\neq k)$. On the other hand, for $j=m+1,\dots,n$,

$$\begin{aligned}\underline{\alpha}_j'(t;\varepsilon)&=-U_j'(t;\varepsilon)=-\left\{\frac{K}{\varepsilon}\sum_{k=1}^{m}U_k(t;\varepsilon)-\frac{L}{\varepsilon}U_j(t;\varepsilon)+\frac{\omega(\varepsilon)}{\varepsilon}+M\right\}\\&=\frac{K}{\varepsilon}\sum_{k=1}^{m}\underline{\alpha}_k(t;\varepsilon)-\frac{L}{\varepsilon}\underline{\alpha}_j(t;\varepsilon)-\frac{\omega(\varepsilon)}{\varepsilon}-M\\&\leqq\frac{1}{\varepsilon}\left\{-K\sum_{k=1}^{m}|u_k|-L\underline{\alpha}_j(t;\varepsilon)-\omega(\varepsilon)\right\}-M\end{aligned}$$

$$<F_j(u_1,\dots, u_m, u_{m+1}, u_{j-1}, \underline{\alpha}_j(t;\varepsilon), u_{j+1},\dots, u_n, t;\varepsilon)\ ,$$

provided $|u_k|\leqq|\underline{\alpha}_k(t;\varepsilon)|=\overline{\alpha}_k(t;\varepsilon)$ $(k\neq j)$.

Thus we see that these functions $\overline{\alpha}_k(t;\varepsilon)$, $\underline{\alpha}_k(t;\varepsilon)$ fulfil the conditions of the Lemma for the functions $F_k(u_1,\dots, u_n, t;\varepsilon)$.

If we suppose the existence of the solution $U_k(t;\varepsilon)$ of the system (6), which has the above-mentioned properties, then the Lemma implies that a solution $u_k(t;\varepsilon)$ of the system (3) satisfying the initial condition

$$u_k(0;\varepsilon)=h_k(\varepsilon)\ \ (k=1, 2,\dots, n)$$

exists over the entire interval $0\leqq t\leqq l$ and fulfils the inequalities

$$|u_k(t;\varepsilon)|<U_k(t;\varepsilon)\ \ (k=1, 2,\dots, n)$$

on the interval $0<t\leqq l$.

Consequently, in order to prove the Theorem 1 it is sufficient to show the existence of the solution $U_k(t;\varepsilon)$ of the linear system (6) fulfilling the following conditions:

U1: The function $U_k(t;\varepsilon)$ exist for sufficiently small $\varepsilon>0$, and are always positive on the interval $0\leqq t\leqq l$.

U2: $|h_k(\varepsilon)|\leqq U_k(0;\varepsilon)$ $(k=1, 2,\dots, n)$.

U3: The functions $U_k(t;\varepsilon)$ converge uniformly to zero on the interval $0\leqq t\leqq l$ as ε tends to zero.

For this purpose, we consider a system of differential equations

$$(7)\qquad \begin{aligned} \frac{dU}{dt}&=mKU+(n-m)KV+\omega(\varepsilon)\ ,\\ \frac{dV}{dt}&=m\frac{K}{\varepsilon}U-\frac{L}{\varepsilon}V+\frac{\omega(\varepsilon)}{\varepsilon}+M\ . \end{aligned}$$

As seen from the forms of the system (6), (7), if $U(t;\varepsilon)$ and $V(t;\varepsilon)$ is a solution of the system (7), then

$$\begin{aligned} U_i(t;\varepsilon)&=U(t;\varepsilon)\quad (i=1, 2,\dots, m),\\ U_j(t;\varepsilon)&=V(t;\varepsilon)\ \ (j=m+1,\dots, n) \end{aligned}$$

is a solution of the system (6). Further, we can see that the general solution $\overline{U}(t;\varepsilon)$, $\overline{V}(t;\varepsilon)$ of the system (7) possesses the following form:

$$(8)\qquad \begin{aligned} &\overline{U}(t;\varepsilon)=C_1(\varepsilon)e^{\lambda_1 t}+C_2(\varepsilon)e^{\lambda_2 t}+\Omega_1(\varepsilon)\ ,\\ &\overline{V}(t;\varepsilon)=\{A+0(\varepsilon)\}C_1(\varepsilon)e^{\lambda_1 t}-\left\{\frac{B}{\varepsilon}+0(1)\right\}C_1(\varepsilon)e^{\lambda_2 t}+\Omega_2(\varepsilon)\ ,\\ &\lambda_1=P+0(\varepsilon)\ ,\ P>0\ ,\ \ \lambda_1=-\frac{L}{\varepsilon}+0(1)\ , \end{aligned}$$

where, $C_1(\varepsilon)$, $C_2(\varepsilon)$ are constants of integration depending on ε, A, B positive constants and $\Omega_1(\varepsilon)$, $\Omega_2(\varepsilon)$ quantities tending to zero with ε.

Therefore, it is easy to prove the existence of the required solution $U_k(t;\varepsilon)$ of the system (6).

Thus Theorem 1 has been proved.

5. Other results. In the foregoing discussion, we have assumed that $h_k(\varepsilon)$ $(k=1,2,\dots,n)$ tend to zero with ε. But even if $h_j(\varepsilon)$ $(j=m+1,\dots,n)$ do not tend to zero with ε, by observing the form (8) of the general solution $\bar{U}(t;\varepsilon)$, $\bar{V}(t;\varepsilon)$ of the system (7), one can prove the following theorem, which we write without the proof.

THEOREM 2. *Let the conditions* C1, C2, C3, C4 *and* C5 *be fulfilled, and given* m *functions of* ε, $h_i(\varepsilon)$ $(i=1,2,\dots,m)$ *which tend to zero with* ε.

Then the non-linear system of differential equations (1) *possesses a solution* $x_k(t;\varepsilon)(k=1,2,\dots,n)$ *over the interval* $0\leqq t\leqq l$, *which satisfies the initial condition*

$$x_i(0;\varepsilon)=X_i(0)+h_i(\varepsilon)\quad(i=1,2,\dots,m)\,,$$
$$x_j(0;\varepsilon)=X_j(0)+h_j\quad(j=m+1,\dots,n)\,,$$

provided that ε *is sufficiently small, where* $h_j(j=m+1,\dots,n)$, *are any numbers such that* $|h_j|<d$. *Furthermore, when* ε *tends to zero, the functions* $x_i(t;\varepsilon)$ $(i=1,2,\dots,m)$ *converge uniformly to the functions* $X_i(t)$ *over the entire interval* $0\leqq t\leqq l$, *and the functions* $x_j(t:\varepsilon)$ $(j=m+1,\dots,n)$ *converge uniformly to* $X_i(t)$ *on any closed interval* $0<l'\leqq t\leqq l$.

By using the Theorem 2, one can easily prove the following.

COROLLARY. *Under the same conditions as in Theorem* 2, *the derivatives of* m *functions* $x_i(t;\varepsilon)$ $(i=1,2,\dots,m)$ *converge uniformly to the derivatives of* $X_i(t)$ *over any closed interval* $0<l'\leqq t\leqq l$ *as* ε *tends to zero.*

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